Pergamon

Int. J. Heat Mass Transfer. Vol. 37, No. 9, pp. 1387–1399, 1994

0017-9310/94 $7.00+0.00

0017-9310(93)E0041-E

Two-dimensional natural convection in an anisotropic and heterogeneous porous medium with internal heat generation

J. J. ROYER
Centre de Recherches Pétrographiques et Géochimiques, 15 rue Nd. des Pauvres, B.P. 20, 54501 Vandoeuvre-lès-Nancy Cedex, France

and

L. FLORES
Instituto de Geofisica, Universidad Nacional Autonoma de Mexico, C.U., 04510 Mexico D.F., Mexico

(*Received* 29 *March* 1993 *and in final form* 15 *November* 1993)

Abstract—A simple dimensionless expression is developed for the natural convection heat transfer. Darcy flow characteristics are assumed for the liquid phase. The solid phase is an anisotropic and heterogeneous saturated porous medium with internal heat generation. Solid characteristics depend on the temperature. The first part of the paper shows that, using a dimensionless formulation, the heat transfer and Darcy equations are similar for two-dimensional problems. The second part reports the numerical results obtained on various cases with a comparison to classical natural convection models.

INTRODUCTION

CONVECTIVE heat transfer in porous media is of fundamental importance to a number of technological applications, such as oil recovery, water supply management in hydrogeology, geothermal exploitation, ground heat storage, radioactive waste management, ground water flow modeling and is also of interest in environmental sciences and geophysics. The general subject of natural convection has received increasing attention since the early experimental observations by Benard [1] and the theoretical work of Rayleigh [2]. Because of its practical importance, particular interest arose during the past decades in heat and mass transfer through natural porous media [3–5] including forced or natural convection [6, 7]. Much of the work on this topic has been concerned with horizontal homogeneous porous layers saturated with liquid phase [8, 9]. Some of these works simplify the anisotropic porous medium to a single anisotropic layer or to a set of stratified homogeneous isotropic or anisotropic porous layers divided by permeable walls, assuming spatial continuity for each layer [10–16]. This approach does not take into account the possible heterogeneity of the studied domain such as porous cavity or deformation processes in sedimentary formations. Other works treat such heterogeneous media at small scale as a homogeneous one at a greater scale using large-scale averaging techniques [17–21]. This method is of great interest to study multi-phase solid media which can be modeled using an homogeneous technique. However, it is not really well adapted to the study of more complicated natural porous media such as geological formations.

The purpose of the present work is to study the natural convection in a rectangular heterogeneous anisotropic saturated porous medium filled with a single phase fluid. Darcy flow characteristics are assumed for the liquid phase. The solid phase is an anisotropic and heterogeneous saturated porous medium with internal heat generation. Solid characteristics depend on the temperature. The first part of the paper uses a dimensionless formulation to simplify the heat and Darcy equations. For two-dimensional problems, the resulting equations are strictly similar and can be solved numerically using the same procedure. Numerical studies are reported and compared to classical approach published in the literature for homogeneous isotropic medium. The effects of boundary conditions on the temperature and flow fields are examined for different cases: (i) isothermally cooled horizontal upper edge, adiabatic vertical edges and constant fixed heat flow at the bottom without fluid exchanges at the boundaries; (ii) isothermally cooled horizontal upper edge, adiabatic vertical edges and constant fixed heat flow at the bottom with con-

NOMENCLATURE

Dimensional parameters

- a_r variation coefficient of the thermal conductivity with the temperature [K^{-1}]
- A^* heat production [$J\ m^{-3}$]
- $\mathbf{e}_1$ unitary vector for the horizontal coordinate system [m]
- $\mathbf{e}_3$ unitary vector for the vertical coordinate system [m]
- g acceleration of gravity [$m\ s^{-2}$]
- H height of the rectangle [m]
- k isotropic mean permeability [m^2]
- $\mathbf{K}$ symmetric tensor of the anisotropic permeability [m^2]
- L width of the rectangle [m]
- $\mathbf{S}$ diagonal matrix associated to the coordinate system (x^*, y^*),

$$\begin{bmatrix} 1/H & 0 \\ 0 & 1/L \end{bmatrix}$$

- t^* time [s]
- T^* temperature [K]
- T_r^* temperature at the reference level, i.e. $T_r^* = (T_1^* + T_2^*)/2$ [K]
- $\mathbf{v}$ filtration velocity [$m\ s^{-1}$]
- x^* horizontal coordinates parallel to L [m]
- z^* vertical coordinates parallel to H [m]
- β_{th} volumic expansion coefficient of the saturating fluid [K^{-1}]
- γ^* variation coefficient of the dynamic viscosity of the saturating fluid with the temperature [K^{-1}]
- ΔT^* temperature difference, $T_2^* - T_1^*$ [K]
- η dynamic viscosity of the fluid [$kg\ m^{-1}\ s^{-1}$]
- η_r dynamic viscosity of the fluid at T_r [$kg\ m^{-1}\ s^{-1}$]
- Λ^* symmetric tensor of the anisotropic thermal conductivity [$W\ m^{-1}\ K^{-1}$]
- ρ_r mass density of the fluid at T_r [$kg\ m^{-3}$]
- $(\rho c)^*$ specific heat content (fluid + solid), at constant pressure [$J\ m^{-3}\ K^{-1}$]
- $(\rho c)_f$ specific heat content of the fluid, at constant pressure [$J\ m^{-3}\ K^{-1}$]
- Φ_h^* vertical heat flow density through horizontal or parallel isothermal limits [$W\ m^{-2}$]
- Φ_v^* horizontal heat flow density through vertical or perpendicular isothermal surfaces [$W\ m^{-2}$].

Dimensionless parameters

- A dimensionless heat production, $(2A^*LH)/(\mathrm{tr}\ \Lambda^*\Delta T^*)$
- e_1 dimensionless unitary vector for the horizontal coordinate system
- e_3 dimensionless unitary vector for the vertical coordinate system
- K dimensionless tensor of permeability in $2D$, $(2/\mathrm{tr}\ \mathbf{K})\cdot\mathbf{K}$
- $\bar{\bar{K}}$ dimensionless tensor of permeability in the dimensionless coordinate system (x, z), $(\alpha\cdot\mathbf{K}\cdot\alpha)$
- Ra^* Darcy modified filtration Rayleigh number for an heterogeneous medium, equation (12)
- t dimensionless time, $(t^*\cdot \mathrm{tr}\ \Lambda)/(2LH(\rho c)^*)$
- T dimensionless temperature, $(T^* - T_r^*)/\Delta T^*$
- $\boldsymbol{v}$ dimensionless flow velocity, $\boldsymbol{v}_{Darcy} - \nabla(\ln \mathrm{tr}\ \Lambda^*)\cdot\Lambda$
- $\boldsymbol{v}_{Darcy}$ dimensionless filtration velocity, components $[v_x, v_z]$
- $|\boldsymbol{v}_{max}|$ maximum dimensionless filtration velocity, $|\boldsymbol{v}_{max}| = \mathrm{Max}\ [v_x^2 + v_z^2]$
- v_x horizontal dimensionless filtration velocity, $(2H(\rho c)_f \mathbf{v}_x)/\mathrm{tr}\ \Lambda^*$
- v_z vertical dimensionless filtration velocity, $(2L(\rho c)_f \mathbf{v}_z)/\mathrm{tr}\ \Lambda^*$
- x dimensionless horizontal x coordinate, x^*/L
- z dimensionless vertical z coordinate, z^*/H
- α dimensionless geometric ratio, H/L
- β dimensionless volumic expansion coefficient of the saturating fluid, $\beta_{th}\ \Delta T$
- γ dimensionless variation coefficient of the dynamic viscosity of the saturating fluid with the temperature, $\gamma^*\Delta T^*$
- ε dimensionless porosity
- Λ dimensionless tensor of thermal conductivity in 2D, $(2/\mathrm{tr}\ \Lambda^*)\cdot\Lambda^*$
- $\bar{\bar{\Lambda}}$ dimensionless tensor of thermal conductivity in the dimensionless coordinate system (x, z), $(\alpha\cdot\Lambda\cdot\alpha)$
- Φ_h dimensionless vertical heat flow density through horizontal or parallel isothermal limits, $(2H\Phi_h^*)/(\mathrm{tr}\ \Lambda^*\Delta T^*)$
- Φ_v dimensionless horizontal heat flow density through vertical or perpendicular isothermal surfaces, $(2L\Phi_v^*)/(\mathrm{tr}\ \Lambda^*\ \Delta T^*)$
- Ψ stream function at stationary conditions.

Superscripts

- t transpose operator
- -1 inverse operator.

Subscripts

- f fluid
- h horizontal or parallel to isothermal limits
- r reference level
- s solid
- v vertical or perpendicular to isothermal surface
- x horizontal component
- z vertical component.

Operators

- **j** rotation matrix $= \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, note that $\mathbf{j}^2 = -\mathbf{I}$ with $\mathbf{I}$ being the identity operator
- α metric tensor associated to the dimensionless coordinate system (x, z), $\begin{bmatrix} \sqrt{\bar{\alpha}} & 0 \\ 0 & 1/\sqrt{\bar{\alpha}} \end{bmatrix}$
- tr trace operator
- det determinant
- $\cdot$ scalar product
- $\otimes$ tensor product
- $\{a_i\}$ summation of terms a_i with respect to i, $\sum_i a_i$
- ∇ 2D nabla operator, $(\partial/\partial x^*, \partial/\partial z^*) =$ **grad**
- **J(f)** Jacobian of vector function **f**
- **H**(f) Hessian of scalar function f
- **D** 2D rotational operator, $(\partial/\partial z^*, -\partial/\partial x^*) = \nabla\,\mathbf{j} =$ **curl**
- Δf Laplacian of a scalar function f.

stant rate fluid exchanges at the vertical boundaries. The study has been conducted for a wide range of parameters; $0 < \alpha < 100$; $Ra \leqslant 600$.

ANALYSIS

The governing equations used to study free convection for incompressible fluids in a rectangular porous domain shown in Fig. 1 is a classical problem including four equations: the heat transfer equation, the Darcy equation (motion equation), the conservative equation and the variation of the fluid characteristics with temperature. Practical investigations commonly assume that the filtration velocity and its gradient are very small causing negligible inertial forces [7, 22]. This approximation leads to the following set of equations for unstationary thermal transfer:

$$\nabla\cdot(\Lambda^*\nabla T^*) = (\rho c)_{\mathrm{f}}\mathbf{v}\cdot\nabla T^* + (\rho c)^*\frac{\partial T^*}{\partial t^*} - A^* \quad (1)$$

$$\nabla p - \rho_{\mathrm{f}}\,\mathbf{g} + \eta\,\mathbf{K}^{-1}\,\mathbf{v} = 0 \quad (2)$$

$$\nabla(\rho_{\mathrm{f}}\mathbf{v}) = 0 \quad (3)$$

$$\rho_{\mathrm{f}} = \rho_{\mathrm{r}}(1-\beta_{\mathrm{th}}(T^*-T_{\mathrm{r}}^*)) \quad (4)$$

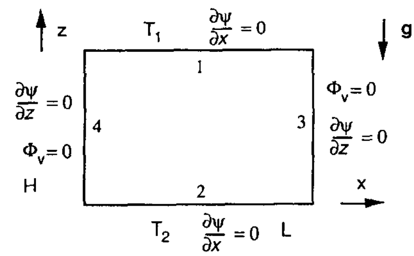


FIG. 1. Confined saturated porous medium being isothermally cooled horizontal upper and lower edges, adiabatic vertical edges without fluid exchanges at the boundaries.

$$\eta = \eta_{\mathrm{r}}(1-\gamma^*(T^*-T_{\mathrm{r}}^*)). \quad (4')$$

It is common to simplify the mass balance and momentum equations assuming an homogeneous medium and that the variations of the fluid density are negligible except in the buoyancy term $\rho\mathbf{g}$ (Boussinesq assumption). However, for large scale hydrothermal systems, some characteristic parameters of the medium (η, Λ^*) may depend on the lithology and on the temperature field while other parameters are less sensitive ($\mathbf{K}$, β_{th}, ρ_{r}). Because of the heterogeneity and of the thermo-dependence of the porous medium, classical convection equations (see ref. [7]) derived for homogeneous porous medium could not be applied. A more elaborate set of equations must be established under given simplified assumptions.

HEAT TRANSFER EQUATION

The heat transfer equation (1) is valid assuming that the difference between the temperature for the solid phase T_{s}^* and for the fluid phase T_{f}^* is negligible. It assumes that the filtration velocity is not too high. Then, the medium can be equivalent to a unique continuum at the average temperature $T^* = T_{\mathrm{f}}^* = T_{\mathrm{s}}^*$. This approach is valid for most common saturated porous geological media such as sedimentary formations, but could be limited for modeling transfers through fractured rocks. The volumic heat capacity of the saturated porous medium $(\rho c)^*$ is assumed to depend on the porosity according to a simple model as following:

$$(\rho c)^* = (1-\varepsilon)(\rho c)_{\mathrm{s}} + \varepsilon\,(\rho c)_{\mathrm{f}}. \quad (5)$$

The thermal conductivity tensor Λ^* decreases generally with increasing temperature following the commonly used relationship [23]:

$$\Lambda^* = \Lambda_{\mathrm{r}}^*/(1+a_{\mathrm{r}}(T^*-T_{\mathrm{r}}^*)) \quad (6)$$

where Λ_r^* is the thermal conductivity at 25°C; a usually ranges from 5×10^{-4} °C^{-1} to 10^{-3} °C^{-1} according to the nature of the porous medium. The thermal conductivity of rocks depends on various parameters (bulk composition, texture, grain size and mineral composition). At 25°C, the thermal conductivity, Λ_0^*, of common rocks increases from 2 W m^{-1} °C^{-1} for migmatites to 3 W m^{-1} °C^{-1} for granites [24]. These variations depend strongly on the quartz content of the rock, according to the experimental relationship established by Koutsikos [25]:

$$\Lambda_0^* = \Lambda_Q^{(0.93\,Qz+0.07)}\,\Lambda_O^{*\,0.93\,(1-Qz)} \tag{7}$$

where Λ_Q^* is the thermal conductivity of quartz (average value 7.7 W m^{-1} °C^{-1}); Λ_O^* is the thermal conductivity of the other minerals (usually ranging around 1.85 W m^{-1} °C^{-1}); and Qz is the quartz content of the rock (in wt.%).

This thermo-dependence of the thermal conductivity makes the heat transfer equation non-linear. This is a serious complication compared to the usual approach which assumes a constant thermal conductivity for the porous medium. However, the effect of the temperature on the thermal properties could not be neglected in problem involving large porous medium such as geothermal systems.

For isotropic medium, the equivalent thermal conductivity may be related experimentally to the porosity according to ref. [8]:

$$\lambda_0 = \lambda_s^{(1-\varepsilon)}\cdot\lambda_f^{\varepsilon} \tag{8}$$

where λ_s and λ_f are, respectively, the thermal conductivity of the solid and of the fluid. This relationship is valid only if the solid and fluid phases are well dispersed and if the contrast between their thermal properties is not too high.

For anisotropic media, the equivalent thermal conductivity tensor Λ^* is a rather more complicated function of the thermal properties of each constituting phase, which can be experimentally determined.

DIMENSIONLESS HEAT TRANSFER EQUATION

Introducing the dimensionless conductivity Λ and noting that:

$$\frac{2}{\operatorname{tr}\Lambda^*}\nabla\cdot(\Lambda\nabla T^*) = \nabla\cdot(\Lambda\nabla T^*)+\nabla\cdot(\ln\operatorname{tr}\Lambda^*)\cdot\Lambda\nabla T^*$$

equation (1) could be rewritten as:

$$\nabla\cdot(\Lambda\nabla T^*) = \frac{2(\rho c)_f}{\operatorname{tr}\Lambda^*}\mathbf{v}\cdot\nabla T^* + \frac{2(\rho c)^*}{\operatorname{tr}\Lambda}\frac{\partial T^*}{\partial t^*} - \frac{2A^*}{\operatorname{tr}\Lambda^*} - \nabla\cdot(\ln\operatorname{tr}\Lambda^*)\cdot\Lambda\nabla T^*. \tag{9}$$

Equation (9) shows that the heterogeneities of the thermal conductivity within space are taken into account as a source term (similar to an internal heat generation) in the second member of the heat equation. Using the dimensionless variables defined in the nomenclature table, equation (9) can be simplified into:

$$\nabla\cdot(\bar{\bar{\Lambda}}\nabla T) = \mathbf{v}\cdot\nabla T+\frac{\partial T}{\partial t}-A$$

$$\boldsymbol{v} = \boldsymbol{v}_{\text{Darcy}}-\nabla(\ln\operatorname{tr}\Lambda^*)\cdot\bar{\bar{\Lambda}} \tag{10}$$

where $\nabla(\ln\operatorname{tr}\Lambda^*)$ denotes the derivative of $\ln\operatorname{tr}\Lambda^*$ with respect to the dimensionless coordinate system (x, z).

DARCY EQUATION

The equation (2), in which some inertial terms have been omitted, is a generalized form of the steady state Darcy relationship. It is valid for describing the motion of the fluid phase in convective systems whose filtration velocity is not too important [7] and for systems such that the ratio between the isotropic mean permeability and the height of the domain (k/H^2) is less than 10^{-3}. Assuming a constant specific heat content for the fluid and solid and a thermo-dependency of the physical characteristic variations of the fluid, ρ and η, equation (2) can be rewritten [22] as:

$$-(1-\gamma T)\boldsymbol{v}_{\text{Darcy}} = \frac{\operatorname{tr}\mathbf{K}}{\operatorname{tr}\Lambda^*}\frac{(\rho c)_f}{\eta_r}\bar{\bar{\boldsymbol{K}}}(\nabla p)^t - Ra^*\left[T-\frac{1}{\beta}\right]\bar{\bar{\boldsymbol{K}}}\cdot\boldsymbol{e}_3 \tag{11}$$

where $\boldsymbol{v}_{\text{Darcy}}$, $\bar{\bar{\boldsymbol{K}}}$, T, β, are the dimensionless variables defined in the nomenclature table. Ra^* is the local filtration Rayleigh number defined for an anisotropic and heterogeneous porous medium as:

$$Ra^* = \frac{\rho_r g(\rho c)_f\beta_{th}\nabla T^* H\operatorname{tr}\mathbf{K}}{\eta_r\operatorname{tr}\Lambda^*}. \tag{12}$$

It explicitly describes the dimensionless criterion for the onset of free hydrothermal convection by measuring the relative influence of the 'driving' force for convection onto the 'stabilizing' effects due to the viscosity of the fluid η_r and to its thermal diffusivity. Homogeneous porous media have been widely studied in the past (see ref. [6] for a review). The condition for the occurrence of thermoconvective cells in such media is that the filtration Rayleigh number is greater than a critical value of $Ra_{cr} = 4\pi^2$. For higher values of Ra, cells become unstable, then convection is turbulent. Using a classical linear theory approach [11] established the theoretical conditions on the appearance of natural convection in a simple anisotropic homogeneous porous layer whose permeability and conductivity tensor being diagonal, respectively, of the form $\mathbf{K} = (k_h, k_h, k_v)$, $\Lambda^* = (\lambda_h, \lambda_h, \lambda_v)$. Using a different definition for the filtration Rayleigh number Ra', the author obtained the following conditions:

$$Ra' = \frac{\rho_r g(\rho c)_f\beta_{th}H\Delta T^* k_h}{\eta_r\lambda_v},$$

$$Ra'_{cr} = \left[\sqrt{\left(\frac{\lambda_h}{\lambda_v}\right)}+\sqrt{\left(\frac{k_h}{k_v}\right)}\right]^2\pi^2,$$

$$L/H = \left[\frac{\lambda_h}{\lambda_v} \cdot \frac{k_h}{k_v}\right]^{1/4} \tag{13}$$

where L/H stands for the characteristic dimension of the convective cells. Incorporating these results in equation (12) for a 2D anisotropic porous layer, one obtains an alternative tensor invariant formula for the critical value for Ra^*:

$$Ra^*_{cr} = \frac{\mathrm{tr}\,\mathbf{K}\,[\mathrm{tr}\,(\Lambda^*\mathbf{K}^{-1})^{1/2}]^2}{\mathrm{tr}\,\Lambda^*}\pi^2 \tag{14}$$

or using dimensionless tensor conductivity or permeability

$$Ra^*_{cr} = [\mathrm{tr}\,(\Lambda \boldsymbol{K}^{-1})^{1/2}]^2\pi^2.$$

DIMENSIONLESS FORMULATION OF THE DARCY EQUATION

Introducing the dimensionless quantities $\boldsymbol{v}_{\mathrm{Darcy}}$, f and g', the Darcy equation can be rewritten in a stream function form as:

$$\boldsymbol{v}_{\mathrm{Darcy}} = (\mathbf{D}\psi)^t$$

$$f = Ra^* \frac{\mathrm{tr}\,\Lambda^*}{\mathrm{tr}\,K}\frac{\eta_r}{(\rho c)_f}\left[T - \frac{1}{\beta}\right],$$

$$g' = \frac{\mathrm{tr}\,\Lambda^*}{\mathrm{tr}\,\mathbf{K}}\frac{\eta_r}{(\rho c)_f}[1-\gamma T]$$

$$(\nabla p)^t = f\boldsymbol{e}_3 - g'\bar{\bar{\boldsymbol{K}}}^{-1}(\mathbf{D}\psi)^t. \tag{15}$$

By cross-differentiation and applying the 2D tensor relationship $\mathrm{tr}\,(\mathbf{D}^t\nabla p) = 0$ reported in Appendix 1, equation (15) can be simplified into the following nonlinear differential equation involving the stream function ψ (see Appendix 2):

$$\mathrm{tr}\,(\mathbf{D}^t(f\boldsymbol{e}_3^t)) = \mathrm{tr}\,(\mathbf{D}^t(g'(\mathbf{D}\psi)\bar{\bar{\boldsymbol{K}}}^{-1})).$$

Further simplifications with respect to the tensorial 2D properties of operators $\mathbf{D}$ and ∇ (see Appendix 2) could be used to derive a more practical simple form for the Darcy equation:

$$\nabla\cdot(\bar{\bar{\boldsymbol{K}}}\nabla\psi) = \boldsymbol{u}\cdot\nabla\psi - S \tag{16}$$

with the following dimensionless quantities

$$\mathbf{u} = -\nabla \ln g\bar{\bar{\boldsymbol{K}}}$$

$$S = (h\nabla \ln g + \nabla h)\cdot\boldsymbol{e}_1$$

$$h = Ra^* \det\bar{\bar{\boldsymbol{K}}}\frac{\left[T-\frac{1}{\beta}\right]}{[1-\gamma T]}$$

$$\mathrm{b}g = \frac{1}{\det\bar{\bar{\boldsymbol{K}}}}\frac{\mathrm{tr}\,\Lambda^*}{\mathrm{tr}\,\boldsymbol{K}}\frac{\eta_r}{(\rho c)_f}[1-\gamma T].$$

The similarity between equation (16) and the heat equation (10) written in a dimensionless form should be noted. The vector quantity $\mathbf{u}$ would be equivalent to the filtration velocity $\mathbf{v}$ in the heat equation, while S would be similar to a source term. This similarity is of great interest when solving numerically the coupled steady state equations of mass and heat transfer because the same procedure can used to solve equations (10) and (16).

EVALUATION OF THE SOURCE TERM S

Logarithmic derivation techniques can be used to evaluate the source term S

$$\nabla \ln g = \nabla \ln(\mathrm{tr}\,\Lambda^*) - \nabla\ln(\mathrm{tr}\,\mathbf{K}) - \nabla\ln(\det\bar{\bar{\boldsymbol{K}}}) - \nabla\ln((\rho c)_f) - \frac{\gamma\nabla T}{(1-\gamma T)}$$

$$\nabla \ln h = \nabla\ln(\det\bar{\bar{\boldsymbol{K}}}) + \nabla\ln(Ra^*) + \frac{\nabla T}{\left(T-\frac{1}{\beta}\right)} + \frac{\gamma\nabla T}{(1-\gamma T)}$$

$$\nabla\ln(Ra^*) = \nabla\ln((\rho c)_f) + \nabla\ln(\mathrm{tr}\,\mathbf{K}) - \nabla\ln(\mathrm{tr}\,\Lambda^*)$$

$$S = \frac{\nabla T}{\left(T-\frac{1}{\beta}\right)}h\cdot\boldsymbol{e}_1 = \frac{Ra^*\det\bar{\bar{\boldsymbol{K}}}}{(1-\gamma T)}\nabla T\cdot\boldsymbol{e}_1. \tag{17}$$

Equation (17) highlights the physical significance of the source term S which depends on the anisotropic properties of the porous medium through $\det\bar{\bar{\boldsymbol{K}}}$ and on the variations of the fluid viscosity with temperature, while the term $\boldsymbol{u}$ depends on the heterogeneity of the medium through the terms $\bar{\bar{\boldsymbol{K}}}$, $\nabla\ln(\mathrm{tr}\,\boldsymbol{K})$, $\nabla\ln(\mathrm{tr}\,\Lambda^*)$, $\nabla\ln(\det\bar{\bar{\boldsymbol{K}}})$. Note that for an isotropic homogeneous medium saturated by a Newtonian fluid with a constant viscosity η independent of the temperature we have $\det\bar{\bar{\boldsymbol{K}}} = 1$, $\nabla\ln(\mathrm{tr}\,\boldsymbol{K}) = \nabla\ln(\mathrm{tr}\,\Lambda^*) = 0$. The source term S in the Darcy equation is reduced to the classical form $S = Ra^*\,\nabla T\cdot\boldsymbol{e}_1$.

CONSERVATIVE EQUATION

The Darcy equation has been derived assuming that the variations of the fluid density are negligible except in the buoyancy term $\rho\mathbf{g}$. This assumption also implies a simplification for the mass balance equation (3) which can be rewritten using the dimensionless filtration velocity $\boldsymbol{v}_{\mathrm{Darcy}}$

$$\nabla\cdot(\mathrm{tr}\,\Lambda^*\,\boldsymbol{v}_{\mathrm{Darcy}}) = 0. \tag{18}$$

In most common applications, the variations of the thermal conductivity within space represented by the term $\mathrm{tr}\,\Lambda^*$ could be neglected in equation (18) compared to the variations of the filtration velocity due to the permeability. For example, the thermal conductivity of common rocks ranges between 1 and 3 W m^{-1} K^{-1} within the temperature interval 25 to 300°C while the permeability ranges between 10^{-20} to 10^{-12} m^2. This approximation gives rises to the simplification of equation (18) into the following:

$$\nabla \cdot \boldsymbol{v}_{\text{Darcy}} = 0. \qquad (19)$$

It should be pointed out that the Darcy velocity $\boldsymbol{v}_{\text{Darcy}}$ in equation (15) derives from the cross derivation of the stream function $(\mathbf{D}\psi)^t$. This property automatically assumes equation (19) because $\nabla \cdot (\mathbf{D}\psi)^t$ is always equal to 0 (Appendix 1).

Note that when the variation of the thermal conductivity within space could not be neglected, developing (18) (Appendix 1) and, incorporating (19) into it, the conservative equation is transformed into:

$$\nabla \operatorname{tr} \Lambda^* \cdot \boldsymbol{v}_{\text{Darcy}} = 0. \qquad (20)$$

Equation (20) forces the scalar product of the dimensionless Darcy velocity by the gradient of the thermal conductivity to be equal to zero. In homogeneous media, this equation is automatically satisfied because the thermal conductivity is constant. At the interface between two homogeneous layers with different thermal conductivity, this implies the Darcy velocity to be parallel to the interface, which is commonly the case when the two layers have different permeability. In other cases, equation (20) has to be included in the governing equations. This case will be no longer discussed in the following.

THE COMPLETE SET OF EQUATIONS

Finally, from the previous developments, the governing equations to study free convection are reduced to the dimensionless coupled equations (10) and (16):

$$\nabla \cdot (\bar{\bar{\Lambda}} \nabla T) = \boldsymbol{v} \cdot \nabla T + \frac{\partial T}{\partial t} - A$$

$$\nabla \cdot (\bar{\bar{\boldsymbol{K}}} \nabla \psi) = \boldsymbol{u} \cdot \nabla \psi - S$$

$$\boldsymbol{v} = \boldsymbol{v}_{\text{Darcy}} - \nabla (\ln \operatorname{tr} \Lambda^*) \cdot \Lambda \quad \boldsymbol{v}_{\text{Darcy}} = (\mathbf{D}\psi)^t$$

$$\boldsymbol{u} = -\nabla \ln g \bar{\bar{\boldsymbol{K}}} \quad S = \frac{Ra^* \det (\bar{\bar{\boldsymbol{K}}})}{(1-\gamma T)} \nabla T \cdot \boldsymbol{e}_1$$

$$g = \frac{1}{\det \bar{\bar{\boldsymbol{K}}}} \frac{\operatorname{tr} \Lambda^*}{\operatorname{tr} \mathbf{K}} \frac{\eta_r}{(\rho c)_f} [1-\gamma T]. \qquad (21)$$

This set of equations is valid for incompressible fluids in an anisotropic and heterogeneous porous medium. At any point of the studied domain, the field pressures can be derived from the integration of the following equation involving the stream function given appropriate boundary conditions:

$$(\nabla p)^t = f \boldsymbol{e}_3 - g' \bar{\bar{\boldsymbol{K}}}^{-1} (\mathbf{D}\psi)^t. \qquad (22)$$

In summary, the analysis concludes with two differential equations reported in (21) that related two unknown functions (T, ψ). In order to close the problem we turn our attention to the boundary conditions along the border of the studied area.

BOUNDARY CONDITIONS

Different boundary conditions can be examined depending on the problem to be solved. Classically, two types of boundary conditions can be defined for each unknown functions (T, ψ). The first one, known as the Neumann conditions, imposes the potential along a boundary, for example the temperature and the stream function are constant. The second type of conditions, known as the Dirichlet conditions, imposes the flow at the boundary, for example the heat flow or the fluid flow through a border.

Isothermally cooled horizontal upper and lower edge, adiabatic vertical edges without fluid exchanges at the boundaries (Fig. 1)

Consider now the Cartesian frame x–y attached to the rectangular domain containing a saturated porous medium. Using the dimensionless system of coordinates (x, z), the boundary conditions can be written as the following:

Upper and lower horizontal boundary conditions 1 and 2:

isothermal cooling:

$$T(x,1) = (T_1^* - T_r^*)/\Delta T^*; \quad T(x,0) = (T_2^* - T_r^*)/\Delta T^*$$

no fluid exchanges:

$$v_z(x,1) = -\frac{\partial \psi}{\partial x}(x,1) = 0;$$

$$v_z(x,) = -\frac{\partial \psi}{\partial x}(x,0) = 0.$$

Upper and lower horizontal boundary conditions 3 and 4:

adiabatic vertical edges:

$$\Phi_v(1,z) = \frac{\partial T^*}{\partial x}(1,z) = 0;$$

$$\Phi_v(0,z) = \frac{\partial T^*}{\partial x}(0,z) = 0$$

no fluid exchanges:

$$v_x(1,z) = -\frac{\partial \psi}{\partial z}(1,z) = 0;$$

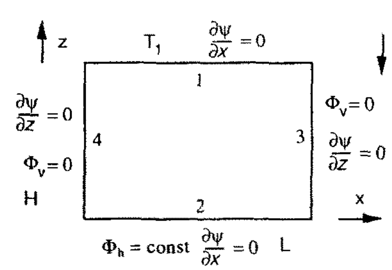


FIG. 2. Confined saturated porous medium being isothermally cooled horizontal upper edge, adiabatic vertical edges and constant fixed heat flow at the bottom without fluid exchanges at the boundaries.

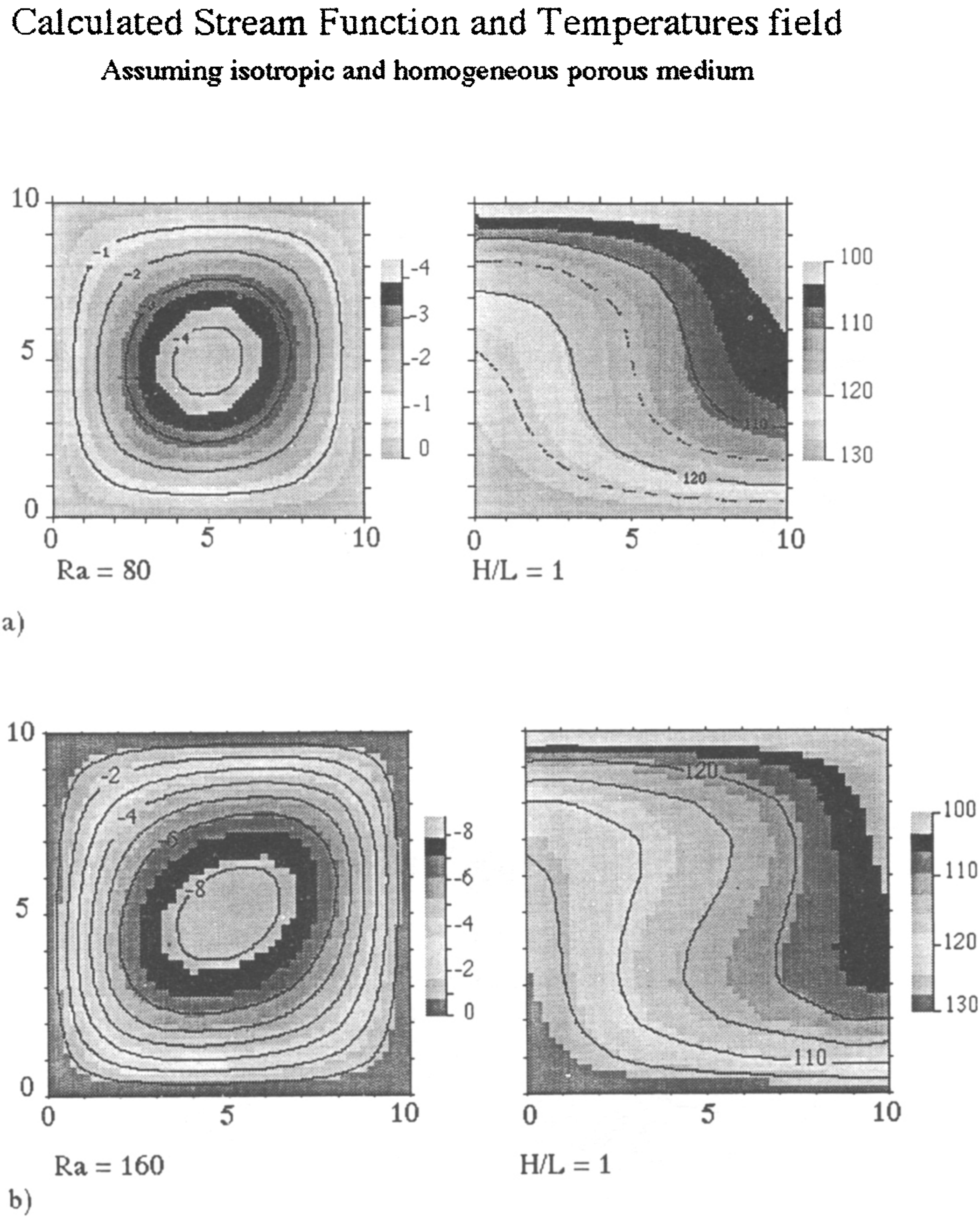


FIG. 3. Steady convective state in an isolated homogeneous isotropic porous medium. Boundary conditions: constant temperature at the horizontal upper (100°C) and lower (130°C) edges, adiabatic vertical edges. (a) Square, $H/L = 1$, $Ra = 80$; (b) square, $H/L = 1$, $Ra = 200$; (c) rectangle, $H/L = 1.8$, $Ra = 200$; (d) rectangle, $H/L = 0.33$, $Ra = 80$.

$$v_x(0, z) = -\frac{\partial \psi}{\partial z}(0, z) = 0.$$

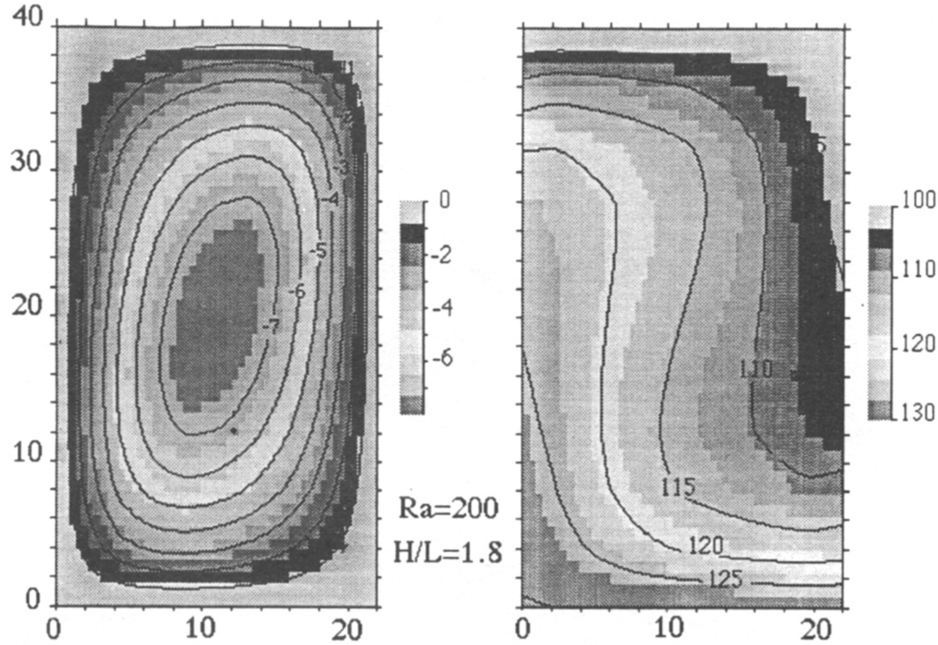
Isothermally cooled horizontal upper edge, adiabatic vertical edges and constant fixed heat flow at the bottom without fluid exchanges at the boundaries (Fig. 2)

The boundary conditions are the same as previously except for the lower horizontal border 2.

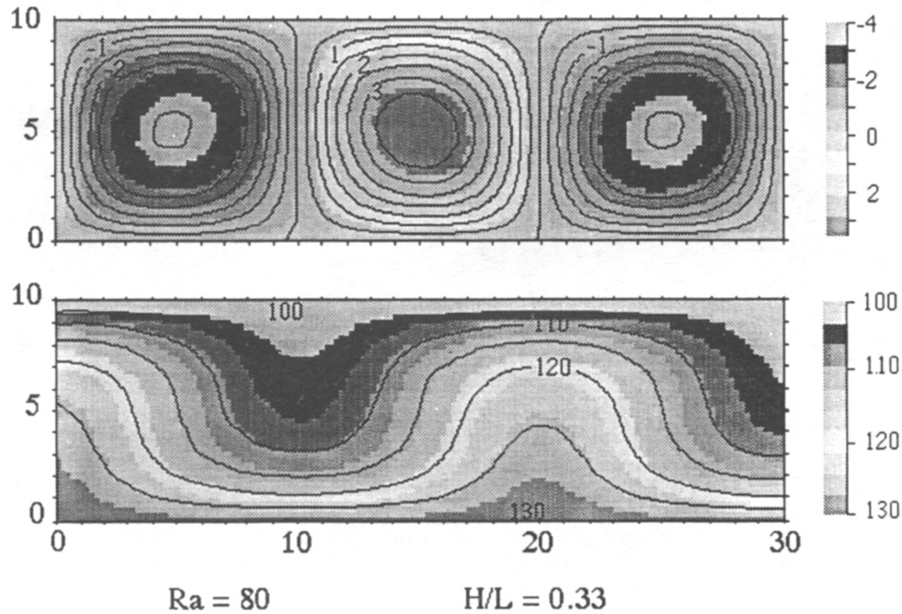
Constant fixed heat flow at the base:

$$\Phi_h(x, 0) = \text{const.}$$

In the following, we will examine the effects of the boundary conditions on the temperature and flow fields for two different cases: (i) isothermally cooled horizontal upper edge, adiabatic vertical edges and constant fixed heat flow at the bottom without fluid exchanges at the boundaries; (ii) isothermally cooled horizontal upper edge, adiabatic vertical edges and



c)



d)

Fig. 3. *Continued.*

constant fixed heat flow at the bottom with constant rate fluid exchanges at the vertical boundaries.

SOLUTION PROCEDURE

The Darcy and heat transport equations are solved by a Gauss–Seidel over-relaxed algorithm using a double iterating method which allows mutual coupling of the equations. Firstly, the heat equation (10) is solved by a finite difference scheme method in order to obtain the stationary temperature field given the conditions imposed at the boundaries.

The numerical resolution starts from an initial state where the saturating fluid phase of the porous medium is motionless but for which a preliminary temperature solution is imposed. A classical biharmonic function is often chosen for the initial temperature field in agreement with theoretical works reported in the literature on convection [6]. Successive instability modes can occur within the porous medium depending on the initial state of the system. Physically, the cellular organization may be described as the superposition of different layers of cells whose dimensions are smaller than the characteristic dimension. The periodicity of the initial solution must be chosen in agreement with these successive instability modes. In this work, the periodicity of the initial solution is a function of the grid node number.

Then, the thermo-dependent terms of the second member of the Darcy equation (16) are evaluated at each grid node as well as the non-linear part of equation (10). Finally, the Darcy equation is numerically solved using a similar procedure as previously used for equation (10). Once the velocity field is estimated, these results are used to evaluate the second member of equation (10).

This flip flop procedure is iteratively applied until the convergence of the solution is reached. We stop computation when the difference between two successive solutions is less than 10^{-3} and 10^{-1} for the temperature and for the stream function respectively.

The discretization scheme used in this study is a classical square finite difference scheme using a five point formula inside the studied domain and a four point formula at the boundaries. The total number of iterations necessary to reach the complete solution depends on the situation but is usually of 50 iterations.

Systems involving large local Rayleigh number could be divergent when the numerical procedure used to invert the linear system is not accurate enough. Stabilization techniques could be applied in these situations, but, in some cases, they could be tricky. The main problem comes from the source term in the second member of the Darcy equation which acts as a perturbation term. When the perturbation is too large, the system of equations becomes divergent. A typical situation is when the permeability of the anisotropic porous medium has a high contrast in magnitude, for instance varying from 10^{-10} to 10^{-16} m^2.

This problem can be solved using the following algorithm: (i) estimate a pseudo solution for the temperature and velocity field at a Rayleigh number $Ra = Ra_0$ for which the convergence of the equations occurs; (ii) increase Ra by a step ΔRa in order to obtain a new set of equations; (iii) solve numerically the set of equations using the previous stationary temperature and velocity field as an initial solution; (iv) apply (ii) to (iii) iteratively till the real Rayleigh number is reached. We have tested this procedure for Rayleigh numbers ranging up to 600 without problems. Note that this algorithm is not valid for transient problems.

NUMERICAL ILLUSTRATIONS

Several trial runs were made to compare the accuracy of the results with those obtained by previous works especially for homogeneous porous media for which extended theoretical, experimental and numerical results have been published [4–8, 12, 13, 16].

Homogeneous isotropic porous media

The first example is an homogeneous porous medium isothermally cooled at the horizontal upper and lower edges, adiabatic vertical edges without fluid exchanges at the boundaries. The stream function and the temperatures field reported in Fig. 3 have been computed for various Rayleigh numbers and geometry. The results obtained by the formulation developed in this paper are fairly similar to the classical behavior of an homogeneous porous medium [6].

When the medium is homogeneous and isotropic, the stationary Darcy velocity and heat equations are simplified into a classical form:

$$\Delta T = \boldsymbol{v}\cdot\nabla T$$

$$\Delta\psi = -Ra^*\,\nabla T\cdot \boldsymbol{e}_1 \qquad (23)$$

with the boundary conditions: $\psi = 0$; $T_1^* = \text{const.}$; $T_2^* = \text{const.}$; $\Phi_3^* = \Phi_4^* = 0$.

Once the system enters a convective state the above boundary conditions make the horizontal temperature variations likely quite independent of the Rayleigh number. Physically, the modulus of the filtration velocity $|\boldsymbol{v}|$ equal to $|\nabla\psi|$ (equation (15)) becomes proportional to the Rayleigh number Ra^*. This simple approximation is not totally true because the two equations are coupled. A more precise theoretical expression can be obtained using a linear stability analysis approach [14]. However, this relationship can be used to test the reliability of the numerical results obtained using the formulation derived from equation (21).

We conduct the numerical experiment for Rayleigh

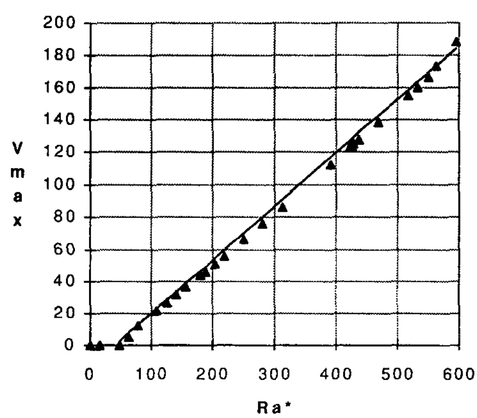


FIG. 4. As expected from the theory, numerical results obtained from a steady convective state in an homogeneous porous medium ($H/L = 1$) show that the maximum dimensionless filtration velocity ($|\boldsymbol{v}_{max}|$) is proportional to the Rayleigh number according to: $|\boldsymbol{v}_{max}| = (Ra^* - 4\pi^2)/3$.

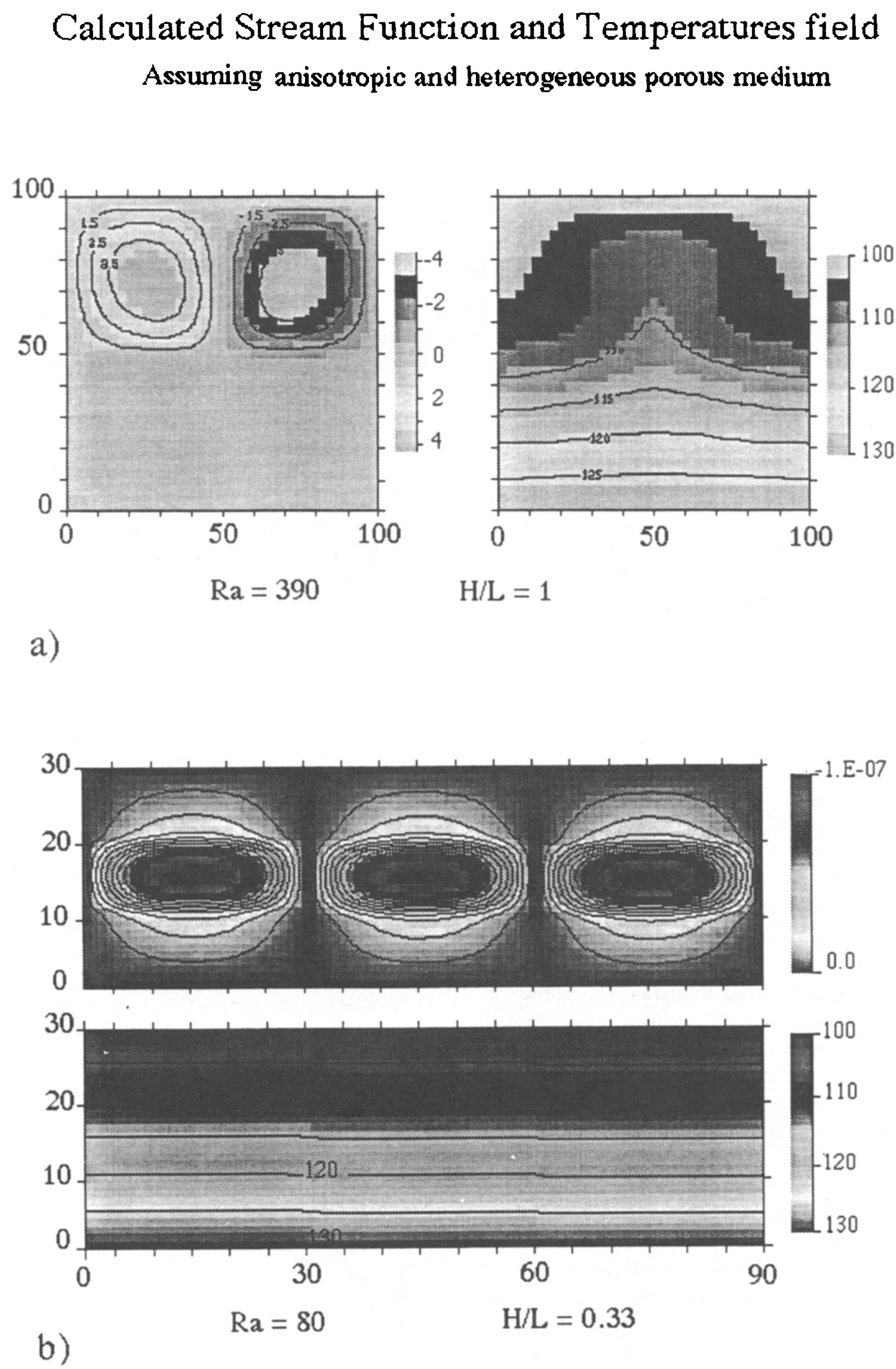


FIG. 5. Steady convective state in an heterogeneous stratiform porous medium constituted by homogeneous isotropic layers. Boundary conditions: constant temperature at the horizontal upper (T_1^*) and lower (T_2^*) edges, adiabatic vertical edges. (a) Two horizontal isolated layers of similar thickness, $H/L = 1$, $T_1^* = 100°C$, $T_2^* = 130°C$, $Ra = 390$; (b) three horizontal isolated layers of similar thickness, $H/L = 0.33$, $T_1^* = 100°C$, $T_2^* = 130°C$, $Ra = 80$; (c) slope porous layer inter-bedded in an impermeable medium with a lateral exchange of fluid of 10^{-10} m^3 s^{-1}, $H/L = 0.33$, $T_1^* = 10°C$, $T_2^* = 230°C$, $Ra = 120$; (d) slope porous layer inter-bedded in an impermeable medium with a lateral exchange of fluid of 10^{-10} m^3 s^{-1}. Boundary conditions: constant temperature at the horizontal upper (T_1^*) and constant fixed heat flow at the bottom (Φ^*), adiabatic vertical edges: $H/L = 0.33$, $T_1^* = 10°C$, $\Phi^* = 100$ mW m^{-2}, $Ra = 280$.

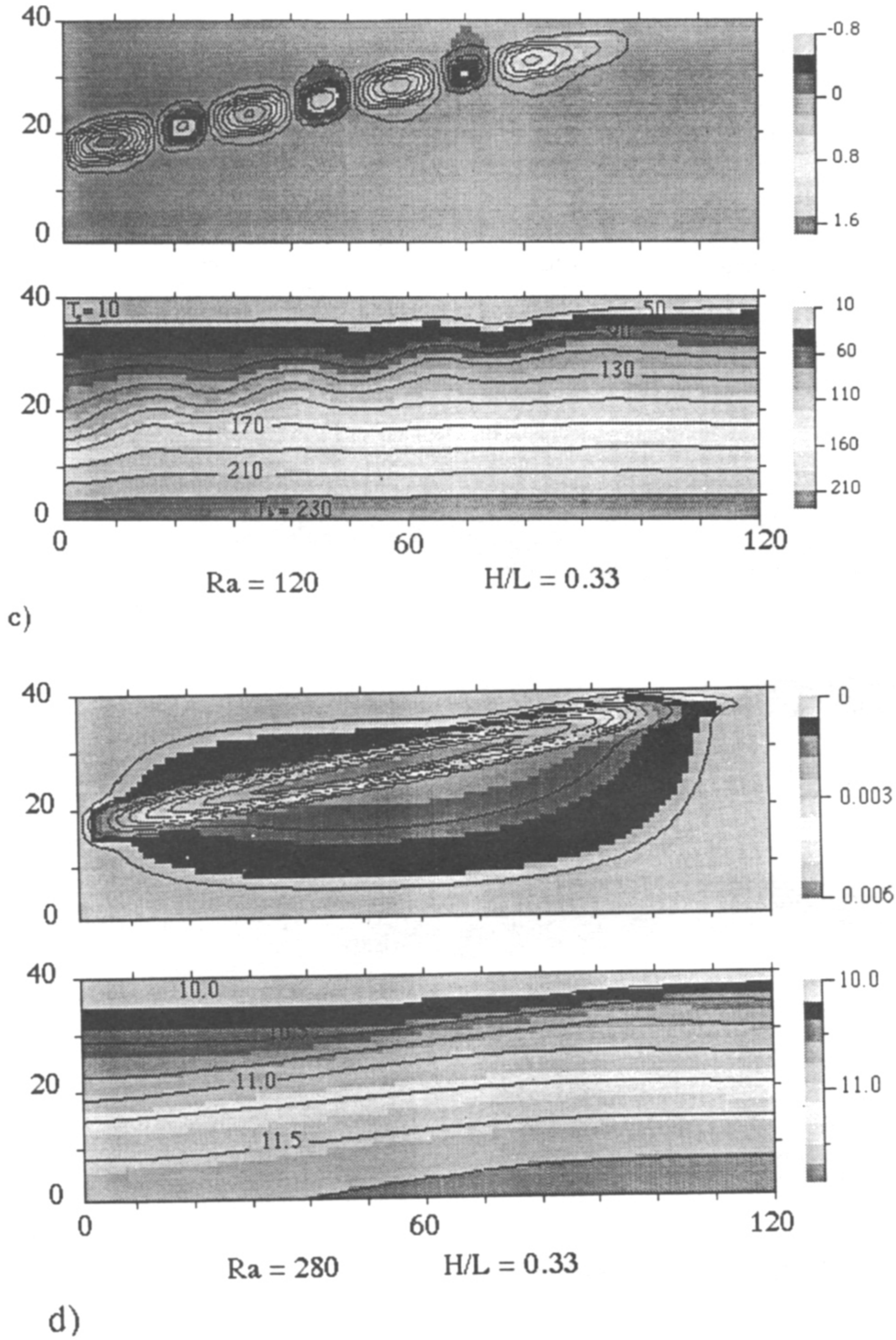


Fig. 5. *Continued.*

numbers ranging from 0 to 600. The results reported in Fig. 4 show a linear relationship between the maximum of the dimensionless Darcy velocity and the Rayleigh number. As expected a non-convective state without fluid movements occurs for Rayleigh number lower than 40. Then, the maximum Darcy velocity is a linear function of Ra^* according to:

$$|\mathbf{v}_{\max}| = \frac{(Ra^* - 4\pi^2)}{3}. \quad (24)$$

Heterogeneous isotropic porous media

The second example is constituted by three horizontal homogeneous isotropic porous layers characterized by a high contrast in permeability and conductivity. Different situations reported in Fig. 5 have been studied varying the geometry and the contrast between the layers. This medium is obviously heterogeneous at large scale. The boundary conditions are similar to the previous example, i.e. a constant temperature at the horizontal upper and lower edges, adiabatic vertical edges, no fluid exchanges across the border.

The first case corresponds to a porous horizontal layer (2) inter-bedded in two impermeable horizontal isotropic layers, respectively, upper (1) and lower (3). The physical characteristics are respectively: $\lambda_1 = 5.2$ W m^{-1} K^{-1}, $K_1 = 10^{-16}$ m^2; $\lambda_2 = 2.5$ W m^{-1} K^{-1}, $K_2 = 10^{-12}$ m^2; $\lambda_3 = 5.2$ W m^{-1} K^{-1}, $K_3 = 10^{-16}$ m^2. The resulting stream function computed from the resolution of the set of equations (21) clearly shows (Fig. 5(b)) preference circulation of fluid in the porous intermediate layer. The number of convective cells is in agreement with the theoretical number predicted by equation (13).

The second situation (Fig. 5(c)) corresponds to an anisotropic porous layer inter-bedded horizontally into an homogeneous impermeable medium: $\lambda_1 = 2.7$ W m^{-1} K^{-1}, $K_1 = 10^{-16}$ m^2; $\lambda_2 = 2.7$ W m^{-1} K^{-1}, $(k_h)_2 = 10^{-12}$ m^2, $(k_v)_2 = 10^{-13}$ m^2; $\lambda_3 = 2.7$ W m^{-1} K^{-1}, $K_3 = 10^{-15}$ m^2.

Finally, Fig. 5(d) reports the results obtained for an homogeneous porous layer with a slope of $\varphi = 15°$ inter-bedded into an homogeneous medium. A fluid flow at a rate of $v_{\mathrm{Darcy}} = 10^{-10}$ m s^{-1} enters the system from above through the lower boundary. The properties of the medium are the following: $\lambda_1 = 5.2$ W m^{-1} K^{-1}, $K_1 = 10^{-15}$ m^2; $\lambda_2 = 2.6$ W m^{-1} K^{-1}, $K_2 = 10^{-12}$ m^2; $\lambda_3 = 5.2$ W m^{-1} K^{-1}, $K_3 = 10^{-15}$ m^2; $v_{\mathrm{Darcy}} = 10^{-10}$ m s^{-1}.

CONCLUSION

The numerical results obtained from the resolution of the set of equations (21) are in good agreement with experimental data and previous results for an isolated box [6, 26]. However, the formulation adopted makes possible the numerical simulation of natural or forced convection in porous media with complex geometry and heterogeneous properties.

Acknowledgements—Funding was provided by the PIRSEM–CNRS (Programme Interdisciplinaire de Recherches sur les Sciences pour l'Energie et les Matières Premières–Centre National de la Recherche Scientifique). The use of the LIAD–CRPG GEOL mapping software was appreciated.

REFERENCES

1. H. Benard, Etude expérimentale du mouvement des liquides propageant de la chaleur par convection. Régime permanent, tourbillons cellulaires, *C.R. Acad. Sci.*, Paris (1900).
2. O. M. Rayleigh, Lord, On convection currents in a horizontal layer of fluid, when the higher temperature is on the under side, *Phil. Mag.* **6**(32), 529–546 (1916).
3. H. S. Carslaw and J. C. Jaeger, *Conduction of Heat in Solids* (2nd Edn), p. 510. Clarendon Press, Oxford (1959).
4. A. Houpeurt, Mécanique des fluides dans les milieux poreux. *Critiques et Recherches*, p. 385. Technip, Paris (1974).
5. J. Crank, *The Mathematics of Diffusion* (2nd Edn), p. 414. Clarendon Press, Oxford (1975).
6. M. A. Combarnous and S. A. Bories, Hydrothermal convection in saturated porous media. In *Advances in Hydroscience*, Vol. 10, pp. 231–307. Academic Press, New York (1976).
7. M. A. Combarnous, Natural convection in porous media and geothermal systems, *6th Int. Heat Transfer Conf., Toronto*, Vol. 6, pp. 45–59 (1978).
8. R. A. Greenkorn, *Flow Phenomena in Porous Media. Fundamentals and Applications in Petroleum, Water and Food Production*, p. 550. Dekker, New York (1983).
9. J. L. Beck, Convection in a box of porous material saturated with fluid, *Phys. Fluids* **15**(8), 1377–1381 (1972).
10. T. Masuoka, Convective currents in a horizontal layer divided by a permeable wall, *Bull. J.S.M.E.* **17**(104), 225–232 (1974).
11. J. F. Epherre, Criterion for appearance of natural convection in an anisotropic porous layer, *Int. Chemical Engng* **17**(4), 615–616 (1977).
12. G. Castinel and M. A. Combarnous, Natural convection in an anisotropic porous layer, *Int. Chemical Engng* **17**(4), 605–617 (1977).
13. C. H. Sondergeld and D. L. Turcotte, An experimental study of two-phase convection in a porous medium with applications to geological problems, *J. Geophys. Res.* **82**(14), 2045–2053 (1977).
14. D. L. Turcotte and G. Shubert, *Geodynamics: Application of Continuum Physics to Geological Problems*, p. 450. Wiley, New York (1982).
15. M. Danis and J. J. Royer, Validity of geothermal heat flow density estimation, *Annal. Geophys.* **3**(3), 357–364 (1985).
16. C. Clauser and H. Villinger, Analysis of conductive and convective heat transfer in a sedimentary basin, demonstration for the Rhinegraben, *Geophys. J. Int.* **100,** 393–414 (1990).
17. A. Bourgeat, Nonlinear homogenization on two phase flow simulation, *Lectures Notes in Pure Applied Mathematics*, Vol. 102, pp. 207–212. Dekker, New York (1985).
18. S. Whitaker, Flow in porous media I: a theoretical derivation of Darcy's Law, *Trans. Porous Media* **1,** 3–25 (1986).
19. S. Whitaker, Flow in porous media II: the governing equations for immiscible, two-phase flow, *Trans. Porous Media* **1,** 105–125 (1986).
20. M. Quintard and S. Whitaker, Ecoulements monophasiques en milieu poreux: effet des hétérogénéités locales, *J. Méca. Théor. Appl.* **6**(5), 691–726 (1987).
21. M. Quintard and S. Whitaker, Two-phase flow in heterogeneous porous media: the method of large-scale averaging, *Trans. Porous Media* **3,** 357–413 (1988).
22. M. Quintard, Stabilité des déplacements miscibles en milieu poreux homogène: injection d'un fluide chaud dans un massif poreux saturé par ce même fluide froid, p. 194. Thèse Doc. d'Etat Es-sciences, Univ. Bordeaux I (1983).
23. S. P. Clark, *Handbook of Physical Constants* (26th Edn), p. 587. Geological Society of America, New York (1966).
24. O. Kappelmeyer and R. Haenel, *Geothermics with Special Reference to Application*, Vol. 1 (4), pp. 1–238. Geoexploration Monographs, Amsterdam (1974).
25. E. Koutsikos, Nouvelles méthodes de détermination de

la conductivité thermique des roches granitoïdes à partir de données minéralogiques et géochimiques. Application aux forages non carottés, p. 196. Thèse Doc. Univ. Pierre et Marie Curie, Paris VI (1985).
26. J. F. Richard, Convection naturelle dans les milieux poreux stratifiés, p. 194. Thèse Doc. Ing., Univ. Bordeaux I (1979).

APPENDIX 1. BASIC TENSOR FORMULA

Notations

f scalar function
$\mathbf{v}$ vector function
$\mathbf{K}$ a matrix

Simple formula

Scalars

$$\nabla f = \operatorname{grad} f = \left(\frac{\partial f}{\partial x^*}, \frac{\partial f}{\partial z^*}\right)$$
$$H(f) = \nabla^t \nabla f = \nabla \otimes \nabla \otimes f = \nabla^2 f$$
$$\operatorname{tr} \mathbf{H}(f) = \Delta f = \nabla \cdot \nabla f$$
$$\mathbf{D} f = (\nabla \mathbf{j}) f = (\nabla f)\mathbf{j}$$
$$\mathbf{D}^t \nabla f = \mathbf{j}^t \nabla^t \nabla f = \mathbf{j}^t \mathbf{H}(f)$$
$$\operatorname{tr}((\nabla f)^t \mathbf{D}) = \operatorname{tr}(\mathbf{D}^t \nabla f) = \operatorname{tr}(\mathbf{j}^t \mathbf{H}(f)) = 0 \quad \text{as } \operatorname{tr} \mathbf{K}^t = \operatorname{tr} \mathbf{K}$$
$$\operatorname{tr}(\nabla^t (\nabla f)\, \mathbf{K}) = \nabla \cdot (\mathbf{K}^t (\nabla f)^t)$$
$$\operatorname{tr}(\nabla^t g \cdot ((\nabla f)\mathbf{K}) = \nabla g \cdot (\mathbf{K}^t (\nabla f)^t)$$

Vectors

$$\nabla \mathbf{v} = \nabla \otimes \mathbf{v} = \frac{\partial v_i^*}{\partial x_j^*} = \mathbf{J}^t(\mathbf{v}) = \mathbf{v}\nabla$$
$$\nabla \cdot \mathbf{v} = \operatorname{div}(\mathbf{v}) = \left\{\frac{\partial v_i}{\partial x_i^*}\right\}$$
$$(\mathbf{v}\nabla \mathbf{j})^t = \mathbf{j}^t \nabla^t \mathbf{v}^t$$
$$\nabla^t(\mathbf{v}^t \mathbf{j}) = (\nabla^t \mathbf{v}^t)\mathbf{j}$$
$$\det(\mathbf{K})\mathbf{j}\mathbf{K}^{-1} = \mathbf{K}\mathbf{j}.$$

Composed formula

$$\nabla(fg) = f\nabla g + g\nabla f$$
$$\nabla(f\mathbf{v}) = \mathbf{v} \otimes \nabla f + f\nabla \mathbf{v} = \mathbf{v}\nabla f + f\nabla \mathbf{v}$$
$$\mathbf{D}^t(f\mathbf{v}^t) = f\mathbf{D}^t\mathbf{v}^t + (\mathbf{D}^t f)\mathbf{v}^t$$
$$(f\mathbf{v}\nabla)^t = \nabla^t(f\mathbf{v}^t) = f\nabla^t\mathbf{v}^t + (\nabla^t f)\mathbf{v}^t$$
$$\nabla \cdot (\mathbf{D} f)^t = 0$$
$$\nabla \cdot (f\mathbf{v}) = \nabla f \cdot \mathbf{v} + f\nabla \cdot \mathbf{v}$$
$$\nabla(\mathbf{K}\mathbf{v}) = \mathbf{K}\nabla\mathbf{v} + \left\{\frac{\partial K_{il}}{\partial x_j} v_l^*\right\}$$
$$\nabla(\mathbf{K}(\nabla f)^t) = \nabla \cdot (\mathbf{K}\nabla f) = \mathbf{K}\nabla \cdot \nabla f + \left\{\frac{\partial K_{il}}{\partial x_j^*}\frac{\partial f}{\partial x_l^*}\right\}$$
$$\nabla^t(\mathbf{v}^t\mathbf{K}) = (\mathbf{K}^t\mathbf{v}\nabla)^t = \nabla^t\mathbf{v}^t \cdot \mathbf{K} + \left\{v_l \frac{\partial K_{lj}}{\partial x_i^*}\right\}$$
$$\nabla^t((\mathbf{D} f)\mathbf{K}) = \nabla^t((\nabla \mathbf{j} f)\mathbf{K}).$$

APPENDIX 2. DERIVING A SIMPLE FORM FOR THE DARCY EQUATION

Applying the $\mathbf{D}$ operator on the right part, then transposing and using the above tensor formula and the symmetrical property of the permeability tensor $\bar{\bar{K}}$, equation (14) can be rewritten:

$$[(\nabla p)^t \mathbf{D}]^t = \mathbf{D}^t \nabla p = \mathbf{j}^t \mathbf{H}(p) = \mathbf{D}^t(f e_3^t) - \mathbf{D}^t(g'(\mathbf{D}\psi)\bar{\bar{K}}^{-1}). \tag{A1}$$

Further, taking the trace of the two members in (A1) and noting that $\operatorname{tr}(\mathbf{j}^t\mathbf{H}(p)) = 0$, it becomes a differential equation with respect to ψ:

$$\operatorname{tr}(\mathbf{D}^t(f e_3^t)) = \operatorname{tr}(\mathbf{D}^t(g'(\mathbf{D}\psi)\bar{\bar{K}}^{-1})). \tag{A2}$$

Note that this technique based on the property of the trace operator is an elegant way to cross differentiated equation (14). However, direct computation would be much more complicated because f and g' are not constant with respect to the coordinate system in the anisotropic non-homogeneous case. Further simplifications can be done in (A2).

Evaluation of $\operatorname{tr}(\mathbf{D}^t(f e_3^t))$ (see Appendix 1)

$$\mathbf{D}^t(f e_3^t) = \mathbf{j}^t \nabla^t(f e_3^t) = \mathbf{j}^t(\nabla^t f \cdot e_3^t) = \mathbf{D}^t f \cdot e_3^t$$

as e_3^t is constant, $\nabla^t e_3^t = 0$

$$\operatorname{tr}(\mathbf{D}^t f \cdot e_3^t) = \operatorname{tr}\left\{\left[\frac{\partial f}{\partial y^*} - \frac{\partial f}{\partial x^*}\right] \otimes [0\ 1]\right\}$$
$$= \operatorname{tr}\begin{bmatrix} 0 & \dfrac{\partial f}{\partial y^*} \\ 0 & -\dfrac{\partial f}{\partial x^*} \end{bmatrix} = -\frac{\partial f}{\partial x^*} = -\nabla f \cdot e_1. \tag{A3}$$

Evaluation of $\operatorname{tr}(\mathbf{D}^t(g'(\mathbf{D}\psi)\bar{\bar{K}}^{-1}))$

$$\mathbf{D}^t(g'(\mathbf{D}\psi)\bar{\bar{K}}^{-1}) = \mathbf{D}^t(g'(\nabla\psi)\mathbf{j}\bar{\bar{K}}^{-1})$$
$$= \mathbf{D}^t(g(\nabla\psi)\bar{\bar{K}}\mathbf{j})$$
$$= \mathbf{j}^t[\nabla^t(g(\nabla\psi)\bar{\bar{K}})]\mathbf{j}$$

with $g' = g \det(\bar{\bar{K}})$ note that $\det(K) = \det(\bar{\bar{K}})$

$$\nabla^t(g(\nabla\psi)\bar{\bar{K}}) = g\nabla^t((\nabla\psi)\bar{\bar{K}}) + \nabla^t g \cdot ((\nabla\psi)\bar{\bar{K}})$$

$$\operatorname{tr}(\mathbf{D}^t(g'(\mathbf{D}\psi)\bar{\bar{K}}^{-1})) = tr(\nabla^t(g(\nabla\psi)\bar{\bar{K}})) \quad \text{as } \operatorname{tr}(\mathbf{j}^t\mathbf{K}\mathbf{j}) = \operatorname{tr}(\mathbf{K})$$
$$= g\operatorname{tr}(\nabla^t((\nabla\psi)\bar{\bar{K}})) + \operatorname{tr}(\nabla^t g \cdot ((\nabla\psi)\bar{\bar{K}}))$$
$$= g\nabla \cdot (\bar{\bar{K}}\nabla\psi) + \nabla g \cdot \bar{\bar{K}}\nabla\psi. \tag{A4}$$

Evaluation of (A2)

Incorporating (A3) and (A4) into (A2) and dividing both member, by g, the Darcy equation is simplified into:

$$\nabla \cdot (\bar{\bar{K}}\nabla\psi) = -\frac{\nabla g}{g} \cdot \bar{\bar{K}}\nabla\psi - \frac{\nabla f}{g} \cdot e_1 \tag{A5}$$

using the following quantities:

$$S = (h\nabla \ln g + \nabla h) \cdot e_1$$
$$h = Ra^* \det \bar{\bar{K}} \frac{\left[T - \frac{1}{\beta}\right]}{[1 - \gamma T]}$$
$$g = \frac{1}{\det \bar{\bar{K}}} \frac{\operatorname{tr} \Lambda^*}{\operatorname{tr} \mathbf{K}} \frac{\eta_r}{(\rho c)_f}[1 - \gamma T].$$

Note that h is defined by $f = hg$, so equation (A5) could be rewritten as:

$$\nabla \cdot (\bar{\bar{K}}\nabla\psi) = -\nabla \ln(g) \cdot \bar{\bar{K}}\nabla\psi - [h\nabla \ln(g) + \nabla h] \cdot e_1.$$

Finally, introducing $\boldsymbol{u} = -\nabla \ln g\, \bar{\bar{K}}$; $S = (h\nabla \ln g + \nabla h) \cdot e_1$, the simple form of the Darcy equation is derived for an anisotropic and heterogeneous porous medium:

$$\nabla \cdot (\bar{\bar{K}}\nabla\psi) = \boldsymbol{u}\nabla\psi - S. \tag{A6}$$